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Pure Appl. Geophys. 169 (2012), 1457–1462
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DOI 10.1007/s00024-011-0419-7

Pure and Applied Geophysics

The Transition from Three-Dimensional Embedding to Two-Dimensional Euler-Lagrange Deformation Tensor of the Second Kind: Variation of Curvature Measures

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Abstract—Based on the Stein formulation of changes of curvature parameters, we describe the changes of Riemann twodimensional manifolds of surface geometries. The three-dimensional left and right Euclidean manifolds with the curvature parameters "geodetic curvature, normal curvature, geodetic torsion" characterize the embedding "Riemann to Euclid" or 2d into 3d. The variation in time of the Euler-Lagrange deformation tensor of the second kind, in short "curvature variation" is studied.

Key words: Continuum mechanics, strain analysis, space-time Geodesy.

1. *Introduction*

Based on plate and shell theory within continuum mechanics we develop formulae for the tensor of change of curvature (TCC) based on surface theory (two-dimensional Riemann manifold) embedded into the ambient space (three-dimensional Euclidean space) at various time instants. Indeed, the curvature tensor is responsible for the detection of vertical displacements as proved by Moghtased-Azar and Grafarend (2009); Grafarend and Voosoghi (2003); Xu and Grafarend (1996). Here we take advantage of the second fundamental forms of surface geometry in the Lagrange-Eulerian versus the Euclidean version within ambient three-dimensional Euclidean space, regularized and graded. We shall present a set of invariants both in the Riemann space as well as in the ambient three-dimensional Euclidean space based on the variational formulation of Stein (1980) of TCC. In terms of differential geometry we take advantage of the Darboux reference frame in terms of {geodetic curvature, normal curvature, geodetic torsion} and the Gauss-Weingarten derivation equations in terms of Gaussian curvature and mean curvature. Our contribution generalizes the work of *Vanicel et al*. Here we enjoy taking reference to modern textbooks on continuum mechanics, for instance, Holzapfel (2000); Hutter and Joehnk (2004); Ernst (1981); Eschenauer and Schnell (1993); Simo and Hughes (1998); Libai and Simmonds (1976); Mushtari and Galimov (1961) and Naghdi (1972). The topic of simultaneous diagonalization of two quadratic forms is treated by Aravind (1988).

2. *Mapping from the Left Riemann Manifold $\mathbb{M}^2_\ell$ to the Right Riemann manifold $\mathbb{M}^2_r$*

Let us start from the left two-dimensional Riemann manifold $\{\mathbb{M}^2_\ell, \mathbf{G}_{\Lambda,\Phi}\}$ as the undeformed manifold embedded in the ambient three-dimensional Euclidean manifold and the right two-dimensional Riemann manifold $\{\mathbb{M}^2_r, \mathbf{g}_{\lambda,\phi}\}$ as the deformed manifold embedded in the ambient three-dimensional Euclidean manifold. In order to represent the two Riemann manifolds in a proper geodetic reference frame, we represent its metric in a Gauss coordinate system which represents the GPS coordinate system: A surface point on the Earth's topography is mapped orthogonally onto the reference ellipsoid by geodetic longitude and geodetic latitude. The third coordinate is defined by the Euclidean distance between the topographic point and the footprint of the reference ellipsoid by an orthogonal projection.

Let us define the reference ellipsoid-of-revolution by the semi-major axis A_1 and the semi-minor axis A_2 leading to the "first eccentricity squared", $E^2 := (A_1^2 - A_2^2)/A_1^2$, at the undeformed manifold as well as

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the reference ellipsoid-of-revolution by the semi-major axis a_1 and the semi-minor axis a_2 leading to the "first eccentricity squared", $e^2 := (a_1^2 - a_2^2)/a_1^2$, at the deformed manifold. We use the ellipsoidal gauge $A_1 = a_1, A_2 = a_2, E^2 = e^2$. In surface geometry we take advantage of the ellipsoidal height representations $H(\Lambda, \Phi)$ and $h(\lambda, \phi)$ introduced in terms of ellipsoidal functions by GRAFAREND and ENGELS (1992) ("amplitude-modified spherical harmonic functions" and "phase-modified spherical harmonic functions").

$$\mathbf{X}(\Lambda, \Phi, H) \in \{\mathbb{R}^3, \mathbf{G}_{\Lambda,\Phi,H}\} \quad \text{or} \quad \mathbf{X}(\Lambda, \Phi, H(\Lambda, \Phi)) \in \{\text{RIEMANN}\}$$

versus

$$\mathbf{x}(\lambda, \phi, h) \in \{\mathbb{R}^3, \mathbf{g}_{\lambda,\phi,h}\} \quad \text{or} \quad \mathbf{x}(\lambda, \phi, h(\lambda, \phi)) \in \{\text{RIEMANN}\}$$

$$\begin{aligned}\mathbf{X}(\Lambda, \Phi, H) &= \mathbf{J}_1 X^1(\Lambda, \Phi, H) + \mathbf{J}_2 X^2(\Lambda, \Phi, H) \\ &\quad + \mathbf{J}_3 X^3(\Lambda, \Phi, H) \\ \mathbf{X}(\Lambda, \Phi) &= [\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3] \\ &\quad \times \begin{bmatrix} \left[\frac{\sim_1}{\sqrt{1-\mathrm{E}^2 \sin^2 \Phi}} + \mathrm{H}(\Lambda, \Phi)\right] \cos\Phi \cos\Lambda \\ \left[\frac{\sim_1}{\sqrt{1-\mathrm{E}^2 \sin^2 \Phi}} + \mathrm{H}(\Lambda, \Phi)\right] \cos\Phi \sin\Lambda \\ \left[\frac{\sim_1(1-\mathrm{E}^2)}{\sqrt{1-\mathrm{E}^2 \sin^2 \Phi}} + \mathrm{H}(\Lambda, \Phi)\right] \sin\Phi \end{bmatrix}\end{aligned}$$

versus

$$\begin{aligned}\mathbf{x}(\lambda, \phi, h) &= \mathbf{j}_1 x^1(\lambda, \phi, h) + \mathbf{j}_2 x^2(\lambda, \phi, h) \\ &\quad + \mathbf{j}_3 x^3(\lambda, \phi, h) \\ \mathbf{x}(\lambda, \phi) &= [\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3] \\ &\quad \times \begin{bmatrix} \left[\frac{\mathrm{a}_1}{\sqrt{1-\mathrm{e}^2 \sin^2 \phi}} + \mathrm{h}(\lambda, \phi)\right] \cos\phi \cos\lambda \\ \left[\frac{\mathrm{a}_1}{\sqrt{1-\mathrm{e}^2 \sin^2 \phi}} + \mathrm{h}(\lambda, \phi)\right] \cos\phi \sin\lambda \\ \left[\frac{\mathrm{a}_1(1-\mathrm{e}^2)}{\sqrt{1-\mathrm{e}^2 \sin^2 \phi}} + \mathrm{h}(\lambda, \phi)\right] \sin\phi \end{bmatrix}\end{aligned}$$

The minimal distance mapping of the topography to the reference ellipsoid has been analyzed in detail by BARTELME and MEISSL (1975); BENNING (1974); FROEHLICH and HANSEN (1976); GRAFAREND and LOHSE (1991); HECK (1987); HEIKKINEN (1982); PAUL (1973); PENEV (1978); PICK (1985); SUENKEL (1976); VINCENTY (1976); VINCENTY (1980).

An important restriction of the *Gauss reference frame* based on two tangent vectors, elements of the *tangent space* $\mathbb{T}_P(\Lambda, \Phi)$ as well as the tangent space $\mathbb{T}_p(\lambda, \phi)$ of surface geometry and the normal space $\mathbb{N}_P(\Lambda, \Phi)$ as well as $\mathbb{N}_p(\lambda, \phi)$, namely,

$$\left\{\mathbf{G}_1 := \frac{\partial \mathbf{X}}{\partial \Lambda}, \mathbf{G}_2 := \frac{\partial \mathbf{X}}{\partial \Phi}\right\} \quad \text{as well as} \quad \left\{\mathbf{g}_1 := \frac{\partial \mathbf{x}}{\partial \lambda}, \mathbf{g}_2 := \frac{\partial \mathbf{x}}{\partial \phi}\right\}$$

versus

$$\mathbf{G}_3 := \frac{\mathbf{G}_1 \times \mathbf{G}_2}{\|\mathbf{G}_1 \times \mathbf{G}_2\|} \quad \text{as well as} \quad \mathbf{g}_3 := \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\|\mathbf{g}_1 \times \mathbf{g}_2\|}$$

is that, due to the parameterization, they become dependent of the ellipsoidal height functions $H(\Lambda, \Phi)$ as well as $h(\lambda, \phi)$ and in consequence also H_Λ, H_Φ as well as h_λ, h_ϕ. A conclusion is that the surface reference frames of type Gauss are no longer orthogonal nor normalized. We summarize by our representation the theory behind the Euler-Lagrange deformation tensor of the second kind as a function of the spatial displacement vector $\mathbf{u}$ and $\mathbf{w}$. We collect results in Tables 1, 2.

For further details of the first, second and third fundamental forms, namely on the curvature matrix and its eigenspace, we refer to KLINGENBERG (1978): The Gauss curvatures $\det \mathbf{K}_\ell$ and $\det \mathbf{K}_r$ are the only curvature functions which do not change sign under the orientation-reversing isometrics or changes of variables. In the case of a differential map from the left to the right differential manifold and vice versa, we summarize various representations of the Euler-Lagrange deformation tensor of the second kind in terms of curvature measures.

3. *Curvature Forms in Gaussian Surface Geometry*

Deformation measures have been developed for changes of placement vectors of a left versus right Riemann manifold embedded into ambient three-dimensional Euclidean manifolds. They are related to changes in

$$\boxed{thefirstfundamentalformofsurfacegeometry} \longleftrightarrow \boxed{thesecondfundamentalformofsurfacegeometry}$$

Table 1

Left versus right curvature matrices

Left Riemann manifold surface curvature Weingarten Map	Right Riemann manifold surface curvature Weingarten Map
$\mathrm{d}\mathbf{G}_3 = -[\,\mathrm{d}\Lambda, \mathrm{d}\Phi\,]\mathbf{H}_\ell\mathbf{G}_\ell^{-1}\begin{bmatrix}\mathbf{G}_1\\ \mathbf{G}_2\end{bmatrix}$	$\mathrm{d}\mathbf{g}_3 = -[\,\mathrm{d}\lambda, \mathrm{d}\phi\,]\mathbf{H}_r\mathbf{G}_r^{-1}\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}$
$\frac{\partial}{\partial[\Lambda,\Phi]} = -\mathbf{H}_\ell\mathbf{G}_\ell^{-1}\begin{bmatrix}\mathbf{G}_1\\ \mathbf{G}_2\end{bmatrix}$	$\frac{\partial}{\partial[\lambda,\phi]} = -\mathbf{H}_r\mathbf{G}_r^{-1}\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}$
Left curvature matrix	Right curvature matrix
$\mathbf{K}_\ell := -\mathbf{H}_\ell\mathbf{G}_\ell^{-1}$	$\mathbf{K}_r := -\mathbf{H}_r\mathbf{G}_r^{-1}$
Left Gauss curvature, left mean curvature (trace, determinant)	Right Gauss curvature, right mean curvature (trace, determinant)
$h_\ell := -\frac{1}{2}\mathrm{tr}\mathbf{K}_\ell = \frac{1}{2}\left[\kappa_\ell^1 + \kappa_\ell^2\right]$	$h_r := -\frac{1}{2}\mathrm{tr}\mathbf{K}_r = \frac{1}{2}\left[\kappa_r^1 + \kappa_r^2\right]$
$k_\ell := \det\mathbf{K}_\ell = \kappa_\ell^1\kappa_\ell^2$	$k_r := \det\mathbf{K}_r = \kappa_r^1\kappa_r^2$
(C.F. Gauss)	(C.F. Gauss)
$\kappa_\ell^1, \kappa_\ell^2 \ldots$ left eigenvalues of $\mathbf{K}_\ell$	$\kappa_r^1, \kappa_r^2 \ldots$ right eigenvalues of $\mathbf{K}_r$

Table 2

Euler-Lagrange deformation tensor of the second kind, Gauss map

Left Euler-Lagrange deformation of the second kind	Right Euler-Lagrange deformation of the second kind
$\mathbb{I}_r - \mathbb{I}_\ell := k_{\lambda,\phi}\,\mathrm{d}q^\lambda\,\mathrm{d}q^\phi - K_{\Lambda,\Phi}\,\mathrm{d}Q^\Lambda\,\mathrm{d}Q^\Phi \quad \text{for} \quad \Lambda, \Phi, \lambda, \phi \in \{1, 2\}$	$\mathbb{I}_r - \mathbb{I}_\ell := k_{\lambda,\phi}\,\mathrm{d}q^\lambda\,\mathrm{d}q^\phi - K_{\Lambda,\Phi}\,\mathrm{d}Q^\Lambda\,\mathrm{d}Q^\Phi \quad \text{for} \quad \Lambda, \Phi, \lambda, \phi \in \{1, 2\}$
$\mathbb{I}_r - \mathbb{I}_\ell := \left(k_{\lambda,\phi}\frac{\partial q^\lambda}{\partial Q^\Lambda}\frac{\partial q^\phi}{\partial Q^\Phi} - K_{\Lambda,\Phi}\right)\mathrm{d}Q^\Lambda\,\mathrm{d}Q^\Phi$	$\mathbb{I}_r - \mathbb{I}_\ell := \left(k_{\lambda,\phi} - K_{\Lambda,\Phi}\frac{\partial q^\Lambda}{\partial Q^\lambda}\frac{\partial q^\Phi}{\partial Q^\phi}\right)\mathrm{d}q^\Lambda\,\mathrm{d}q^\phi$
$\mathbb{I}_r - \mathbb{I}_\ell := \kappa_{\Lambda,\Phi}\,\mathrm{d}Q^\Lambda\,\mathrm{d}Q^\Phi$	$\mathbb{I}_r - \mathbb{I}_\ell := \kappa_{\lambda,\phi}\,\mathrm{d}q^\Lambda\,\mathrm{d}q^\phi$
subject to	*subject to*
$\kappa_{\Lambda,\Phi} := k_{\lambda,\phi}\frac{\partial q^\lambda}{\partial Q^\Lambda}\frac{\partial q^\phi}{\partial Q^\Phi} - K_{\Lambda,\Phi}$	$\kappa_{\lambda,\phi} := k_{\lambda,\phi} - K_{\Lambda,\Phi}\frac{\partial q^\Lambda}{\partial Q^\lambda}\frac{\partial q^\Phi}{\partial Q^\phi}$

Table 3

Euler-Lagrange deformation tensor of the second kind, Gauss map

Left representation	Right representation
$\kappa_{\Lambda,\Phi} = k_{\lambda,\phi}\frac{\partial q^\lambda}{\partial Q^\Lambda}\frac{\partial q^\phi}{\partial Q^\Phi} - K_{\Lambda,\Phi} \quad (2)$	$\kappa_{\lambda,\phi} = k_{\lambda,\phi} - K_{\Lambda,\Phi}\frac{\partial Q^\Lambda}{\partial q^\lambda}\frac{\partial Q^\Phi}{\partial q^\phi} \quad (6)$
$\kappa_{\Lambda,\Phi} =$	$\kappa_{\lambda,\phi} :=$
$:= -\dfrac{\partial q^\lambda}{\partial Q^\Lambda}\dfrac{\partial q^\phi}{\partial Q^\Phi}\left\langle\dfrac{\partial \mathbf{g}_3}{\partial q^\lambda}, \mathbf{g}\phi\right\rangle - K_{\Lambda,\Phi} \quad (3)$	$:= k_{\lambda,\phi} + \dfrac{\partial Q^\Lambda}{\partial q^\lambda}\dfrac{\partial Q^\Phi}{\partial q^\phi}\left\langle\dfrac{\partial \mathbf{G}_3}{\partial Q^\lambda}, \mathbf{G}\phi\right\rangle \quad (7)$
$\kappa_{\Lambda,\Phi} = -\frac{\partial q^\lambda}{\partial Q^\Lambda}\frac{\partial q^\phi}{\partial Q^\Phi}\cdot\cdot\left\langle\frac{\partial(\mathbf{w}+\mathbf{G}_3}{\partial q^\lambda}, \frac{\partial(\mathbf{X}+\mathbf{u})}{\partial q^\phi}\right\rangle - K_{\Lambda,\Phi} \quad (4)$	$\kappa_{\lambda,\phi} = k_{\lambda,\phi} - \frac{\partial Q^\Lambda}{\partial q^\lambda}\frac{\partial Q^\Phi}{\partial q^\phi}\cdot\cdot\left\langle\frac{\partial(\mathbf{w}-\mathbf{g}_3}{\partial Q^\Lambda}, \frac{\partial(\mathbf{x}-\mathbf{u})}{\partial Q^\Phi}\right\rangle \quad (8)$
$\kappa_{\Lambda,\Phi} =$	$\kappa_{\lambda,\phi} =$
$= -\langle\mathbf{w}_{1_\Lambda}, \mathbf{G}\Phi\rangle - \langle\mathbf{w}_{1_\Lambda}, \mathbf{u}_\Phi\rangle - -\langle\mathbf{u}_{1_\Phi}, \mathbf{G}_{3,\Lambda}\rangle \quad (5)$	$= -\langle\mathbf{w}_{1_\lambda}, \mathbf{g}_3\rangle - \langle\mathbf{w}_{1_\lambda}, \mathbf{u}\phi\rangle - -\langle\mathbf{u}_{1_\phi}, \mathbf{g}_{3,\lambda}\rangle \quad (9)$

Here we concentrate on the tensor of change of the curvature (TCC) in terms of left and right coordinates as developed by PIETRASZKIEWICZ (1977); [STEIN (1980). Their theory forms the basis of plate and shell theory. In analogy to the Euler-Lagrange deformation of the first kind, they introduced the deformation tensor of the second kind or the TCC as functions of the spatial displacement of surface normal vectors in the left and right Riemann manifold, namely

$$\mathbf{w} := \mathbf{g}_3 - \mathbf{G}_3 \tag{1}$$

([Stein (1980)])

In our geodetic concept of TCC we have documented already that TCC measures the changes of vertical displacements of manifolds. In a set of formulae we collect the results of TCC theory, the Stein equations. We use the inner product notation of partial derivatives of the normal vector bundle based on $\mathbf{w}, \mathbf{g}_3$ and $\mathbf{G}_3$. Finally we are left with the Euler-Lagrange deformation tensor of the second kind as a function of spatial displacement vectors $\mathbf{u}$ and $\mathbf{w}$.

We have derived the Mean Curvature $h := \frac{1}{2}\mathrm{tr}\mathbf{K}$ and the Gauss Curvature $k := \det \mathbf{K}$ from the eigenvalue–eigenvector representation

$$\begin{gathered} \kappa\mathbf{I}\mathbf{v} = \mathbf{H}\mathbf{G}^{-1}\mathbf{v} \;\Leftrightarrow \det(\kappa\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}) = 0 \\ \Leftrightarrow \det(\mathbf{H} - \kappa\mathbf{G}) = 0,\; \mathbf{G} \text{ positive-definite} \\ \det(\kappa\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}) = \kappa^2 - 2h\kappa + k = 0 \\ \kappa_{1,2} = h \pm \sqrt{h^2 - k^2}. \end{gathered} \tag{10}$$

We are confronted with the problem of determining simultaneously diagonalize the pair of matrices $-\mathbf{H}_\ell\mathbf{G}_\ell^{-1}$ and $-\mathbf{H}_r\mathbf{G}_r^{-1}$ forming left and right curvature. Unfortunately the matrices of curvature "left and right" are not positive-definite, neither semi-definite, in general. Indeed, we cannot apply our result from Theorem 1.1 of Grafarend and Krumm (2006) page 1:

> *If* $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $\mathbf{B} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix such that the product $\mathbf{A}\mathbf{B}^{-1}$ exists, then there exists a non-singular matrix $\mathbf{X}$ such that both following matrices are diagonal matrices, where $\mathbf{I}_n$ is the n-dimensional unique matrix.

$$\mathbf{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{11}$$

$$\mathbf{X}^{\mathrm{T}}\mathbf{B}\mathbf{X} = \mathbf{I}_n = \mathrm{diag}(1, \ldots, 1). \tag{12}$$

We come back to this problem elsewhere.

4. Curvature Forms in the Darboux Reference Frame in the Three-Dimensional Euclidean Space

So far we have analyzed the Gauss curvature concept and introduced the Euler-Lagrange deformation tensor of the second kind according to surface geometry and left/right Riemann manifolds of change of curvature. Here we present the theory of the Darboux frame of reference in terms of

$$geodeticcurvature, normalcurvature, geodetictorsion$$

and study its changes in terms of the embedding into the ambient three-dimensional Euclidean space.

Let us first define the orthonormal triad or 3-leg of a surface curve of type Darboux, a special CARTAN frame of reference.

$$\begin{aligned} \mathbf{d}_1 &:= \frac{\partial \mathbf{x}}{\partial \lambda} / \|\mathbf{x}_\lambda\| \\ \mathbf{d}_2 &:= \mathbf{g}_3 \times \mathbf{d}_1 \\ \mathbf{d}_3 &:= \mathbf{g}_3 \end{aligned}$$

Next we study its embedding into the three-dimensional Euclidean space. We start from the representation

$$\mathrm{d}\mathbf{d} = \Omega\mathbf{d}, \quad \Omega = -\Omega^{\mathrm{T}}, \quad \Omega := (\mathrm{d}\mathbf{R})\mathbf{R}^{\mathrm{T}}$$

The antisymmetric connection matrix Ω has, therefore, only three components called

$$geodeticcurvature, normalcurvature, geodetictorsion$$

$$\begin{bmatrix} \mathrm{d}\mathbf{d}_1 \\ \mathrm{d}\mathbf{d}_2 \\ \mathrm{d}\mathbf{d}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_1^2 & \omega_1^3 \\ -\omega_1^2 & 0 & \omega_2^3 \\ -\omega_1^3 & -\omega_2^3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}$$

$$\begin{aligned} \kappa_g &:= \frac{\omega_1^2}{\sigma^1} \quad \text{``\textit{geodetic curvature}''} \\ \kappa_n &:= \frac{\omega_1^3}{\sigma^1} \quad \text{``\textit{normal curvature}''} \\ \tau_g &:= \frac{\omega_2^3}{\sigma^1} \quad \text{``\textit{geodetic torsion}''} \end{aligned}$$

$$\begin{aligned} \mathrm{d}\mathbf{d}_1 &= \sigma^1\kappa_g\mathbf{d}_2 + \sigma^1\kappa_n\mathbf{d}_3 \\ \mathrm{d}\mathbf{d}_2 &= -\sigma^1\kappa_g\mathbf{d}_1 + \sigma^1\tau_g\mathbf{d}_3 \\ \mathrm{d}\mathbf{d}_3 &= -\sigma^1\kappa_n\mathbf{d}_1 - \sigma^1\tau_g\mathbf{d}_2 \end{aligned}$$

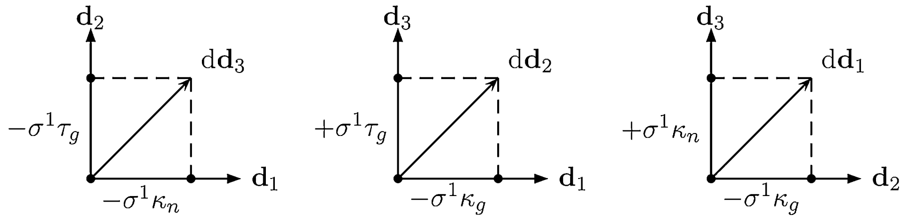

Figure 1
Projections of $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ on the tangential plane as well as on the horizontal plane

$$\textit{subject to } \sigma^1 := \|\mathbf{x}_\lambda\|$$

Geodetic curvature κ_g measures the change of the tangent vector $\mathbf{d}_1$ and of the tangent vector $\mathbf{d}_2$ located on the local tangent plane. The normal curvature κ_n takes care of the change of the tangent vector $\mathbf{d}_1$ as well as the normal vector $\mathbf{d}_3$ while geodetic torsion τ_g measures the change of the surface normal vector projected on the tangent vector $\mathbf{d}_1$.

Figure 1 illustrates the various projections of $\mathrm{d}\mathbf{d} = \Omega\mathbf{d}$.

Basic is the transformation of the Darboux frame of reference to a local Frenet frame of reference.

Lemma 1 *(J. Meusnier: Memoir sur la courbure des surfaces, Mem. des Savants etr. 10 (1785) 504)*

$$\kappa_g = \kappa_1 \sin\theta, \quad \kappa_n = \kappa_2 \cos\theta, \quad \tau_g = \kappa_2 + \frac{\mathrm{d}\theta}{\sigma^1}$$

$$\kappa_g^2 + \kappa_n^2 = \kappa_1^2$$

(κ_g, κ_n, τ_g) *are the elements of the connection of the Darboux 3-leg*, (κ_1,κ_2) are the descriptive elements of the connection of the Frenet 3-leg. Of course, Frenet 1 = Darboux 1 holds. θ denotes the angle between the base vectors Frenet 2 (principal normal vector) and the surface normal vector Darboux 3. How do we transfer this result to the Euler-Lagrange deformation tensor of surface geometry, a surface embedded into a three-dimensional Euclidean space? We consider the Stein equations $\mathbf{w} = \mathbf{g}_3 - \mathbf{G}_3$ and its derivatives as a model of changes of curvature parameters, namely Eqs. (1)–(10). They describe the changes of Riemann two-dimensional manifolds of surface geometries. The three-dimensional Euclidean manifolds with the curvature parameters "geodetic curvature, normal curvature, geodetic torsion" characterize the embedding Riemann→Euclid or 2d into 3d. Their variation in time is the basis of the Euler-Lagrange deformation tensor of the second kind, in short "curvature variation".

5. Conclusion

We have first represented the Euler-Lagrange deformation tensor of the second kind, the surface curvature tensor deformation in the left and right forms, as well as its transformation "left-right" by formula (6)–(9). In contrast, we next introduced the curvature variational forms in the Darboux reference frame in the three-dimensional Euclidean space, the embedding space illustrated in Fig. 1. Based on the Meusnier Lemma we presented the transformation from three-dimensional Euclidean space, namely its curvature parameters {geodetic curvature, normal curvature, geodetic torsion}, into the eigenvalues of the Gauss curvature tensor in surface geometry. In addition, there appears the angle θ between base vector Frenet 2 (principal normal vector) and the surface normal vector Darboux 3. We leave the question open to transform the variational equations for the Euler-Lagrange tensor of the second kind from the left to the right form and vice versa. We have two forms of the Meusnier Lemma, one for the left manifold and one for the right manifold. The variation in time of the manifolds, left and right, has to be calculated by

$$\begin{aligned}\kappa_g(\text{left}) - \kappa_g(\text{right}) = f\{&\kappa_1(\text{left}), \kappa_2(\text{left}), \theta(\text{left}),\\ &\kappa_1(\text{right}), \kappa_2(\text{right}), \theta(\text{right})\},\\ &\kappa_n(\text{left}) - \kappa_n(\text{right}) = f\{\ \},\\ &\tau_g(\text{left}) - \tau_g(\text{right}) = f\{\ \},\\ &\quad\text{subject to the condition}\\ &\quad\kappa_g^2 + \kappa_n^2 = \kappa_1^2(\text{left}) \quad \text{and}\\ &\quad\kappa_g^2 + \kappa_n^2 = \kappa_1(\text{right}).\end{aligned}$$

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(Received November 11, 2010, revised March 29, 2011, accepted May 11, 2011, Published online November 12, 2011)